The Ninth Grade Math Competition Class
Divisibility Rules
Anthony Wang

1. What is the least number greater than 9000 that is divisible by 11?

$$9000 \equiv -9+0-0+0 \equiv -9 \pmod{11}$$
$$\equiv 2 \pmod{11}$$

$$\Rightarrow 9000 \equiv 2 \pmod{11}$$

$$9009 \equiv 11 \equiv 0 \pmod{11}$$

2. Find A such that $3A6$ is a multiple of 9.

$9 \mid \underline{3A6} \Rightarrow 3+A+6 \equiv 0 \pmod 9$

$9+A \equiv 0 \pmod 9$

$A \equiv 0 \pmod 9$

$\boxed{A = 0, 9}$

3. Find the ordered pairs of digits (A, B) such that $67A7B$ is a multiple of 225.

$$\underline{67A7B} \equiv 0 \pmod{225}$$

$$225 = 3^2 \cdot 5^2$$

$$225 = 9 \cdot 25$$

$$\underline{\overset{75}{7B}} \equiv 0 \pmod{25}$$

$B = 5$

$$\underline{67A75}$$

$$6+7+A+7+5 \equiv 0 \pmod 9$$

$$25 + A \equiv 0 \pmod 9$$

$$7 + A \equiv 0 \pmod 9$$

$$\Rightarrow A = 2$$

4. Find the value of the digit D if $47D4$ leaves a remainder of 2 when divided by 33.

$$47D4 \equiv 2 \pmod{33}$$

$$47D2 \equiv 0 \pmod{33}$$

$$33 = 3 \cdot 11$$

$$4+7+D+2 \equiv 0 \pmod 3$$

$$13+D \equiv 0 \pmod 3$$

$$1+D \equiv 0 \pmod 3$$

$D = 2, 5, 8$

$$-4+7-D+2 \equiv 0 \pmod{11}$$

$$5-D \equiv 0 \pmod{11}$$

$D = 5$

$$\boxed{D=5}$$

5. A four-digit number uses each of the digits 1, 2, 3 and 4 exactly once. Find the probability that the number is a multiple of 4.

1 2 $\underline{3\,4}$

34, 43 | 12 ✓
~~13~~
14 ×

13, 31 | ~~21~~, ~~23~~, 24 ✓

14, 41 | ~~31~~, 32 ✓, 34 ×

~~41~~
42 ×
~~43~~

$$\frac{6}{24} = \boxed{\frac{1}{4}}$$

6. Find the ordered pair of digits (M, N) such that $52MN5$ is a multiple of 1125.

$$\underline{52MN5} \equiv 0 \pmod{1125}$$

$$1125 = 9 \cdot 125$$

$$5+2+M+N+5 \equiv 0 \pmod{9}$$

$$12+M+N \equiv 0 \pmod{9}$$

$$3+M+N \equiv 0 \pmod{9}$$

$$\underline{MN5} \equiv 0 \pmod{125}$$

$125,\ \cancel{250},\ 375,\ \cancel{500},\ 625,\ \cancel{750},\ 875$

$3+1+2 \equiv 6 \pmod{9}$ $\quad 3+3+7 \equiv 4$ $\ \ldots\ $ $3+6+2 \equiv 2$ $\ \ldots\ $ $\boxed{3+8+7 \equiv 18 \equiv 0 \pmod{9}}$

$\to M=8$

$N=7$

7. For all integer values of $n \geq 2$, k will divide $n^3 - n$. What is the greatest possible integer value of k?

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$$

$n=2$ $\quad 1 \cdot 2 \cdot 3$ $\qquad k = 2 \cdot 3 = \boxed{6}$

$n=3$ $\quad 2 \cdot 3 \cdot 4$

$n=4$ $\quad 3 \cdot 4 \cdot 5$

$n=5$ $\quad 4 \cdot 5 \cdot 6$

8. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

$15 = 3 \cdot 5$

$80 \equiv 8 + 0 \not\equiv 0 \pmod{3}$

$880 \equiv 16 + 0 \not\equiv 0 \pmod{3}$

$\underline{8}\,\underline{8}\,\underline{8}\ \underline{0}$

$8 + 8 + 8 + 0 \equiv 0 \pmod{3}$