The Ninth Grade Math Competition Class
Modular Arithmetic
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a, b

1. The remainders when two natural numbers are divided by 12 are 5 and 9. (a) Find the remainder when their product is divided by 12. (b) Find the reminder when their product is divided by 4.

(a) $a \bmod 12 = 5$ $\quad a \equiv 5 \pmod{12}$

$b \bmod 12 = 9$ $\quad b \equiv 9 \pmod{12}$

$ab \equiv 45 \pmod{12}$

$ab \bmod 12 = 45 \bmod 12 = 9$

(b)

$ab = 12q + 9$

$ab \bmod 4 = (12q + 9) \bmod 4$

$= 1$

2. Is $21^{100} - 12^{100}$ a multiple of 11?

$$\left(21^{100} - 12^{100}\right) \bmod 11 = 0 \qquad \text{YES}$$

$$1 - 1 = 0$$

$$21^{100} \bmod 11 = (21 \bmod 11)^{100} \bmod 11$$

$$= 10^{100} \bmod 11$$

$$21^{100} \equiv 10^{100} \equiv (-1)^{100} \pmod{11}$$

$$\equiv 1 \pmod{11}$$

$$12^{100} \equiv 1^{100} \equiv 1 \pmod{11}$$

3. Find the remainder when $24^{50} - 15^{50}$ is divided by 13.

$(24^{50} - 15^{50}) \bmod 13$

$24^{50} \bmod 13 = (-2)^{50} \bmod 13$

$24 \equiv 11 \equiv -2 \pmod{13}$

$24 \bmod 13 = 11 \bmod 13 = -2 \bmod 13$

$15^{50} \bmod 13 = 2^{50} \bmod 13$

$15 \bmod 13 = 2$

$\left((-2)^{50} - 2^{50}\right) \bmod 13$

$\left(\underbrace{(-1)^{50}}_{=1} 2^{50} - 2^{50}\right) \bmod 13 =$

$(1 \cdot 2^{50} - 2^{50}) \bmod 13 = 0$

4. Find the tens and units digits of 7^{2006}.

$$
\left.\begin{array}{l}
7^0 \to 01 \\
7^1 \to 07 \\
7^2 \to 49 \\
7^3 \to 43
\end{array}\right.
$$

$$
\left.\begin{array}{l}
7^4 \to 01 \\
7^5 \to 07 \\
7^6 \to 49 \\
7^7 \to 43
\end{array}\right.
$$

$$7^{2006} \to 7^{2006 \bmod 4}$$

$$\to 7^2 \to 49$$

5. Find the remainder when $1^2 + 2^2 + 3^2 + \cdots + 99^2$ is divided by ~~13~~ 9.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 \cdots + 99^2 \bmod 9$$

$$\left.\begin{aligned} 1^2 &\equiv 1 \\ 2^2 &\equiv 4 \\ 3^2 &\equiv 0 \\ 4^2 &\equiv 7 \end{aligned}\right\} 12 \qquad (\text{mod } 9)$$

$$\left.\begin{aligned} 5^2 &\equiv 7 \\ 6^2 &\equiv 0 \\ 7^2 &\equiv 4 \\ 8^2 &\equiv 1 \end{aligned}\right\} 12$$

$$9^2 \equiv 0$$

$$10^2 \equiv 1^2 \equiv 1$$

$$11^2 \equiv 2^2 \equiv 4$$

$$24 \cdot 11 \bmod 9$$

$$6 \cdot 2 \bmod 9 \equiv 3$$

6. Find the remainder when $9^{42} - 5^{42}$ is divided by 7.

$$9^{42} - 5^{42} \bmod 7$$

$$9^{42} \equiv 2^{42} \pmod 7$$

$$5^{42} \equiv (-2)^{42} \pmod 7$$

$$\left(2^{42} - (-2)^{42}\right) \bmod 7$$

$$\left(2^{42} - (-1)^{42} 2^{42}\right) \bmod 7 = 0$$

$$\uparrow$$

$$1$$

7. Find the remainder when 7^{255} is divided by 11.

$7^0 \equiv 1 \pmod{11}$

$7^1 \equiv 7$

$7^2 \equiv 5$

$7^3 \equiv 2$

$7^4 \equiv 3$

$7^5 \equiv 10$

$7^6 \equiv 4$

$7^7 \equiv 6$

$7^8 \equiv 9$

$7^9 \equiv 8$

$7^{10} \equiv 1$

$7^{255} \equiv 7^{255 \bmod 10} \equiv 7^5 \equiv 10$

$7^{11} \equiv 7 \pmod{11}$

$7^{10} \equiv 1 \pmod{11}$

8. Find the last two digits of 99^{2005}.

$99^{2005} \bmod 100$

$\equiv (-1)^{2005} \pmod{100} \equiv -1 \equiv 99$

9. A natural number n, has a unit digit of A when expressed in base 12. Find the remainder when n^2 is divided by 6.

$n \bmod 12 = 10$

$n = 12q + 10$

$n^2 \equiv (\cancel{12q} + 10)^2 \bmod 6$

$\equiv 10^2 \equiv 100 \pmod 6$

$\equiv 4$