The Ninth Grade Math Competition Class
Quadratic Formula and Polynomial
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$\frac{1}{2}$ $\qquad\qquad$ $\frac{3}{2}+\frac{1}{2}=2$

1. Find the value of x if x is positive and $x - 1$ is the reciprocal of $x + \frac{1}{2}$.

$$\frac{1}{x-1} = x + \frac{1}{2}$$

$$1 = \left(x+\frac{1}{2}\right)(x-1)$$

$$1 = x^2 + \frac{1}{2}x - x - \frac{1}{2}$$

$$2 = 2x^2 + x - 2x - 1$$

$$0 = 2x^2 - x - 3$$

$$0 = (2x-3)(x+1)$$

$$x = \cancel{-1},\ \frac{3}{2}$$

2. It is given that one root of $2x^2 + rx + s = 0$, with r and s real numbers, is $3 + 2i$. Find s.

$3-2i$

$(3+2i)(3-2i) = \frac{s}{2}$

$2(3+2i)(3-2i) = s$

$2(9+4) = \boxed{26 = s}$

3. Find all values of k such that $x^2 + kx + 27 = 0$ has two distinct real solutions for x.

$b^2 - 4ac > 0$

$k^2 - 4 \cdot 1 \cdot 27 > 0$

$k^2 > 108$

$$\boxed{\begin{array}{l} k > 6\sqrt{3} \\ k < -6\sqrt{3} \end{array}}$$

4. Find all real solutions to $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

$$a^b = 1$$

$$1^b = 1 \qquad x^2 - 5x + 5 = 1$$

$$x^2 - 5x + 4 = 0$$

$$(-1)^{b \text{ even}} = 1 \qquad (x-4)(x-1) = 0 \quad \boxed{x = 1, 4}$$

$$a^0 = 1$$

$$x^2 - 5x + 5 = -1$$

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0 \quad \boxed{x = 2, 3}$$

$$x^2 - 9x + 20 = 0$$

$$(x-4)(x-5) = 0$$

$$\boxed{x = 4, 5}$$

5. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

$$x^2 + y = y^2 + x$$

$$x^2 + y - y^2 - x = 0$$

$$(x-y)(x+y) - (x-y) = 0$$

$$(x-y)(x+y-1) = 0$$

$$x - y = 0 \qquad x^2 + x = 12$$

$$x^2 + x - 12 = 0$$

$$(x+4)(x-3) = 0$$

$$\boxed{\begin{matrix} x = -4 \\ y = -4 \end{matrix}} \qquad \boxed{\begin{matrix} x = 3 \\ y = 3 \end{matrix}}$$

$$x + y - 1 = 0 \qquad x^2 + 1 - x = 12$$

$$y = 1 - x \qquad x^2 - x - 11 = 0$$

$$x = \frac{1 \pm \sqrt{1+44}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$$

$$\boxed{x = \frac{1+3\sqrt{5}}{2} \quad y = \frac{1-3\sqrt{5}}{2}}$$

$$\boxed{x = \frac{1-3\sqrt{5}}{2} \quad y = \frac{1+3\sqrt{5}}{2}}$$

6. Find all values of m for which the zeros of $2x^2 - mx - 8$ differ by $m - 1$.

$$x = \frac{m \pm \sqrt{m^2-64}}{4}$$

$$\frac{m+\sqrt{m^2-64}}{4} - \frac{m-\sqrt{m^2-64}}{4} = \frac{\sqrt{m^2-64}}{2} = m-1$$

$$\frac{m^2-64}{4} = m^2-2m+1$$

$$m^2-64 = 4m^2-8m+4$$

$$0 = 3m^2-8m-60$$

$$0 = (3m+10)(m-6)$$

$$\boxed{m = 6, -\frac{10}{3}}$$

7. A polynomial of degree four with leading coefficient 1 and integer coefficients has two zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

$(A)\frac{1+i\sqrt{11}}{2}$ $(B)\frac{1+i}{2}$ $(C)\frac{1}{2}+i$ $(D)1+\frac{i}{2}$ $(E)\frac{1+i\sqrt{13}}{2}$

$$(x-r)(x-s)\cancel{(x-t)(x-u)}$$

$$(x-r)(x-s)=x^2+ax+b$$

$$r+s=-a$$

$$rs=b$$

$$\frac{1+i\sqrt{11}}{2} \quad \frac{1-i\sqrt{11}}{2}$$

$$\frac{1+11}{4}=3$$

$$\frac{1+i}{2} \quad \frac{1-i}{2}$$

$$\frac{1+1}{4}=\frac{1}{2}$$

8. Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.

$\cancel{x^{2001}} + \dots - \cancel{x^{2001}}$ deg-2000 polynomial

$$a^{2001} + \left(\frac{1}{2} - a\right)^{2001} = 0$$

$$\left(\frac{1}{2} - a\right)^{2001} + \underbrace{a^{2001}}_{\frac{1}{2} - \left(\frac{1}{2} - a\right)} = 0$$

sum of each pair $\quad a + \frac{1}{2} - a = \frac{1}{2}$

number of pairs $\quad \frac{2000}{2} = 1000$

$\Rightarrow \frac{1}{2} \cdot 1000 = 500$

$f(1) = 1^4 + a + b + c$

9. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $-2,\ -3,\ 5$. Find the value of $a + b + c$.

$f(x) =$

$$-\frac{0}{1} = r - 2 - 3 + 5$$

$$0 = r - 2 - 3 + 5$$

$$r = 0$$

$$f(x) = x(x+2)(x+3)(x-5) = 0$$

$$f(1) = 1(3)(4)(-4) = -48 = 1 + a + b + c$$

$$a + b + c = -49$$

$ax^2+bx+c=0 \quad r,s$

$r+s=-\frac{b}{a}$

$rs=\frac{c}{a}$

10. One root of the quadratic $x^2+bx+c=0$ is $1-3i$. If b and c are real numbers, then what are b and c?

$1+3i$

$((x-1)-3i)((x-1)+3i) = (x-1)^2+9$

$= x^2-2x+1+9$

$= x^2-2x+10$

$-b = 1-3i+1+3i = 2$

$c = (1-3i)(1+3i) = 1+9 = 10$

$a+b+c=-3$

$f(-3)=-27+27-12-11=-23$

$f(x)=$

11. Suppose the roots of $x^3+3x^2+4x-11=0$ are a, b and c, and the roots of $x^3+rx^2+sx+t=0$ are $a+b$, $b+c$, and $c+a$, find the value of t.

$-t=(a+b)(b+c)(c+a)$ $\quad a+b+c$

$-t=(-3-c)(-3-a)(-3-b)$

$t=-(-3-c)(-3-a)(-3-b)=-f(-3)=23$

$f(x)=(x-c)(x-a)(x-b)$

12. Let a, b, and c be the roots of $f(x) = x^3 - 3x^2 + 1$.

$f(a) = 0$
$f(b) = 0$
$f(c) = 0$

- Find a polynomial whose roots are $a + 3$, $b + 3$ and $c + 3$. $g(x)$
- Find a polynomial whose roots are $\frac{1}{a+3}$, $\frac{1}{b+3}$, and $\frac{1}{c+3}$. $h(x)$
- Compute $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} = \frac{45}{53}$
- Find a polynomial whose roots are a^2, b^2 and c^2. $l(x)$

$g(a+3) = 0$
$g(b+3) = 0$
$g(c+3) = 0$

$g(x) = f(x-3)$

$g(a+3) = f(a+3-3) = f(a)$

$g(x) = (x-3)^3 - 3(x-3)^2 + 1$

$= x^3 - 12x^2 + 45x - 53$

$h\left(\frac{1}{a+3}\right) = h\left(\frac{1}{b+3}\right) = h\left(\frac{1}{c+3}\right) = 0$

$h(x) = g\left(\frac{1}{x}\right)$

$h\left(\frac{1}{a+3}\right) = g(a+3) = 0$

$h(x) = \frac{1}{x^3} - \frac{12}{x^2} + \frac{45}{x} - 53$

$x^3 h(x) = -53x^3 + 45x^2 - 12x + 1$

$l(a^2) = l(b^2) = l(c^2) = 0$

$l(x) = f(\sqrt{x})$

$l(a^2) = f(\sqrt{a^2}) = f(a) = 0$

$l(x) = x^{\frac{3}{2}} - 3x + 1 = x^{\frac{3}{2}} - (3x - 1)$

$(x^{\frac{3}{2}} + (3x-1))\, l(x) = (x^{\frac{3}{2}} - (3x-1))(x^{\frac{3}{2}} + (3x-1))$

$= x^3 - 9x^2 + 6x - 1$

$x = r_1, r_2, r_3 \qquad r_1 + r_2 + r_3$

13. The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum.

$2^{-2} \quad 2^{333x}$

$\frac{1}{4} 2^{333x} + 4 \cdot 2^{111x} = 2 \cdot 2^{222x} + 1$

$\quad y^3 \qquad\qquad y \qquad\qquad y^2$

$y = 2^{111x}$

$\log_2 y = 111x$

$\frac{\log_2 y}{111} = x$

$\frac{1}{4} y^3 + 4y = 2y^2 + 1$

$y^3 - 8y^2 + 16y - 4 = 0$

s_1, s_2, s_3

$\frac{\log_2 y}{111} = r_1, r_2, r_3$

$\frac{\log_2 s_1}{111} + \frac{\log_2 s_2}{111} + \frac{\log_2 s_3}{111}$

$\frac{1}{111}(\log_2 s_1 s_2 s_3)$

$\frac{(\log_2 4)}{111} = \frac{2}{111}$

14. If $P(x)$ is a polynomial in x such that for all x, $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9).P(x)$, compute the sum of coefficients of $P(x)$.

$1 + 23 - 18 - 24 + 108 = (1 - 3 - 2 + 9)P(1)$

$P(1) = 18$

15. The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

$$x + \frac{1}{x} = \sqrt{5}$$

$$x^2 + 2 + \frac{1}{x^2} = 5$$

$$x^2 + \frac{1}{x^2} = 3$$

$$x^4 + 2 + \frac{1}{x^4} = 9$$

$$x^4 + \frac{1}{x^4} = 7$$

$$x^7\left(x^4 - 7 + \frac{1}{x^4}\right)$$

$$x^7(7-7) = 0$$

16. All the roots of the polynomial $x^6 - 10z^5 + Az^4 + Bz^3 + cZ^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 2 & 2 \end{array}$$

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 10$$

$$r_1 r_2 r_3 r_4 r_5 r_6 = 16$$

$$1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

$$-B = r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_2 r_5 + r_1 r_2 r_6$$

$$r_2 r_3 r_4 + \_\ \_\ \_$$

$$8\binom{4}{3} = 32 \qquad 2\binom{4}{1}\binom{2}{2} = 8$$

$$4\binom{4}{2}\binom{2}{1} = 48 \qquad 32 + 48 + 8 = 88$$

$$B = -88$$