The Ninth Grade Math Competition Class
Prime Factorization 1
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1. What is the smallest positive integer N such that the value $7 + 30N$ is not a prime number?

$N=1$ $7+30N=37$ is prime

$N=2$ $7+30N=67$ is prime

$N=3$ $7+30N=97$ is prime

$N=4$ $7+30N=127$ is prime

$N=5$ $7+30N=157$ is prime

$N=6$ $7+30N=187$ is not prime

$187=11\cdot 17$

2. The product of a set of positive integers is 140. What is their least possible sum?

$140 = 2 \cdot 70 \Rightarrow 2 + 70 = 72$

$140 = 2 \cdot 2 \cdot 35 \Rightarrow 2 + 2 + 35 = 39$

$140 = 2 \cdot 2 \cdot 5 \cdot 7 \Rightarrow 2 + 2 + 5 + 7 = 16$

3. Find the greatest natural number that must be a divisor of any common multiple of 14, 26 and 66.

$\text{lcm}(14,26,66) \mid k \cdot \text{lcm}(14,26,66)$

divides

$14 = 2^1 \cdot 7^1$

$26 = 2^1 \cdot 13^1$

$66 = 2^1 \cdot 3^1 \cdot 11^1$

$\text{lcm}(14,26,66) = 2^1 \cdot 3^1 \cdot \underbrace{7^1 \cdot 11^1 \cdot 13^1}_{1001}$

$6 \cdot 1001 = 6006$

4. The product of any two of the possible integers 30, 72 and N is divisible by the third. What is the smallest possible value of N?

$2^1 \cdot 3^1 \cdot 5^1 \qquad 2^3 \cdot 3^2$

$N \mid 30 \cdot 72 = 2^4 \cdot 3^3 \cdot 5^1$

$30 \mid 72 \cdot N \Rightarrow 2^1 \cdot 3^1 5^1 \mid 2^3 \cdot 3^2 \cdot N$

$72 \mid 30 \cdot N \Rightarrow 2^3 \cdot 3^2 \mid 2^1 \cdot 3^1 \cdot 5^1 \cdot N$

$N = 2^2 \cdot 3^1 \cdot 5^1 = 60$

5. How many divisors of 5400 are not multiples of any perfect square greater than 1?

$$5400 = 54 \cdot 100$$
$$= 2^1 \cdot 3^3 \cdot 2^2 \cdot 5^2$$
$$= 2^3 \cdot 3^3 \cdot 5^2$$

$d \mid 5400$

$$d = \begin{array}{ccc} 2^0 & 3^0 & 5^0 \\ 2^1 & 3^1 & 5^1 \\ \cancel{2^2} & \cancel{3^2} & \cancel{5^2} \\ \cancel{2^3} & \cancel{3^3} & \end{array}$$

$$2 \cdot 2 \cdot 2 = 8$$

6. How many of positive divisors of 45000 themselves have exactly 12 positive divisor?

$$45000 = 45 \cdot 1000$$
$$= 3^2 \cdot 5 \cdot 2^3 \cdot 5^3$$
$$= 2^3 \cdot 3^2 \cdot 5^4$$

$d \mid 45000 \qquad t(d) = 12$

$d = p^{11} \qquad 12$

0

$d = p^5 q^1 \qquad 6 \cdot 2$

0

$\overset{p}{2} \cdot \overset{q}{2} = 4 \qquad d = p^3 \cdot q^2 \qquad 4 \cdot 3$

$\dfrac{\overset{p}{3} \cdot \overset{q}{2} \cdot \overset{r}{1}}{2} = \dfrac{6}{2} = 3 \qquad d = p^2 \cdot q^1 \cdot r^1 \qquad 3 \cdot 2 \cdot 2$

$2^2 \cdot 3^1 \cdot 5^1$

$2^2 \cdot 5^1 \cdot 3^1$

$4 + 3 = 7$

7. If m has 10 positive divisors, n has 6 positive divisors, and $gcd(m, n) = 1$, how many positive divisors does mn have?

$$t(m)t(n) = t(mn)$$

$$10 \cdot 6 = 60$$

8. If n has exactly 7 positive divisors, how many positive divisors does n^2 have?

$n = p^6$

$n^2 = p^{12} \Rightarrow 13$

9. How many of the positive divisors of 168 are even?

$$168 = 2 \cdot 84$$
$$= 2 \cdot 2^2 \cdot 3 \cdot 7$$
$$= 2^3 \cdot 3 \cdot 7$$

$$\frac{168}{2} = 2^2 \cdot 3^1 \cdot 7^1 \qquad (2+1)(1+1)(1+1) = 12$$

$d \mid 168$ $\quad d$ is even

$$d = \begin{array}{ccc} \cancel{2^0} & 3^0 & 5^0 \\ 2^1 & 3^1 & 5^2 \\ 2^2 & & \\ 2^3 & & \end{array}$$

$$3 \cdot 2 \cdot 2 = 12$$

$\Longleftrightarrow$

10. Show that any positive perfect square has an odd number of positive divisors?

$12^2 = 144 = 2^4 \cdot 3^2$

$(2^4 \cdot 3^2)^{\frac{1}{2}} = 2^2 \cdot 3^1$

$n = p_1^{2c_1} p_2^{2c_2} \cdots p_k^{2c_k}$

$t(n) = (2c_1+1)(2c_2+1) \cdots (2c_k+1)$