The Ninth Grade Math Competition Class
Divisors
Anthony Wang

1. Find the product of the positive divisors of 2400 that are multiples of 6.

$$2400 = 24 \cdot 100$$
$$= 2^3 \cdot 3^1 \cdot 2^2 \cdot 5^2$$
$$= 2^5 \cdot 3^1 \cdot 5^2$$

$d \mid 2400 \quad 6 \mid d$

$$400 = 2^4 \cdot 5^2$$

$(4+1)(2+1) = 15$

$e \mid 400$

$6e \mid 2400$

product of div. of 400

$= 400^{\frac{15}{2}}$

$400^{\frac{15}{2}} \cdot 6^{15}$

$\boxed{20^{15}\, 6^{15}}$

2. Find the product of the divisors of 3200 that are perfect squares.

$$3200 = 32 \cdot 100$$
$$= 2^5 \cdot 2^2 \cdot 5^2$$
$$= 2^7 \cdot 5^2 \qquad 2^6 \cdot 5^2$$

$$2^0 \quad 5^0$$
$$d^2 \mid 3200 \qquad 2^2 \quad 5^2$$
$$2^4$$

$$d \mid \sqrt{2^6 \cdot 5^2} = 40 \quad 2^6$$

$$40 = 2^3 \cdot 5^1$$

$$40^{\frac{8}{2}} = 40^4$$

$$(40^4)^2 = 40^8$$

3. A proper divisor of a number is a divisor of the number that is not the number itself. What is the smallest positive integer that is less than the sum of its positive proper divisors?

abundant #s	divisors	sum of proper divisors
2	1, 2	1
3	1, 3	1
4	1, 2, 4	3
5	1, 5	1
6	1, 2, 3, 6	6
7	1, 7	1
8	1, 2, 4, 8	7
9	1, 3, 9	4
10	1, 2, 5, 10	8
11	1, 11	1
12	1, 2, 3, 4, 6, 12	16

$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

4. How many positive cubes divide $3! \cdot 5! \cdot 7!$?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

$$n! = n \cdot (n-1)(n-2) \cdot \ldots \cdot 1$$

$$3! = 2^1 \cdot 3^1$$
$$5! = 2^3 \cdot 3^1 \cdot 5^1$$
$$7! = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$
$$n = 3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$$

2^0 3^0 5^0 7^0
2^3 3^3
2^6

$$d^3 \mid n$$

$$3 \cdot 2 \cdot 1 \cdot 1 = 6$$

5. How many of positive divisors of 3200 are not multiples of any perfect square greater than 1?

$$3200 = 2^7 \cdot 5^2$$

$2^0 \quad 5^0$

$2^1 \quad 5^1$

$\cancel{2^2} \quad \cancel{5^2}$

$2 \cdot 2 = 4$

6. How many positive integers have exactly three proper divisors, each of which is less than 50?

four divisors

$n = p^3 \quad 1, p, p^2 < 50 \quad p = 2, 3, 5, 7 \quad 4$

$n = p^1 q^1 \quad 1, p, q < 50$

$\frac{15 \cdot 14}{2} = 105$

$p = 2, 3, 5, 7, 11, 13, 17, 19$
$23, 29, 31, 37, 41, 43, 47$

15 primes < 50

$4 + 105 = 109$

7. Jan is thinking of a positive integer. Her integer has exactly 16 positive divisors, two of which are 12 and 15. What is Jan's number?

$12 \mid n$ $\quad 12 = 2^2 \cdot 3^1$

$15 \mid n$ $\quad 15 = 3^1 \cdot 5^1$

$n = 2^3 \cdot 3^1 \cdot 5^1 = 120$

$4 \cdot 2 \cdot 2 = 16$

8. What is the sum of all positive integers less than 100 that have exactly 12 divisors?

$n = \cancel{p^{11}}$

$= p^5 \cdot q^1$ $\quad 6 \cdot 2$ $\quad 2^5 \cdot 3^1 = 96$

$= p^3 \cdot q^2$ $\quad 4 \cdot 3$ $\quad 2^3 \cdot 3^2 = 72$

$2^3 \cdot 5^2 = \cancel{126}$

$= p^2 \cdot q^1 \cdot r^1$ $\quad 3^3 \cdot 2^2 = \cancel{108}$

$2^2 \cdot 3^1 \cdot 5^1 = 60$

$2^2 \cdot 3^1 \cdot 7^1 = 84$

$2^2 \cdot 3^1 \cdot 11^1 = \cancel{132}$

$3^2 \cdot 2^1 \cdot 5^1 = 90$

$\boxed{402}$

9. Dentoe p_k be the k^{th} prime number. Show that $p_1 p_2 \cdots p_n + 1$ cannot be the perfect square of an integer.

$$p_1 p_2 p_3 \cdots p_n + 1 = a^2$$

$$p_1 p_2 p_3 \cdots p_n = a^2 - 1 = \underbrace{(a+1)}^{2c+1}\underbrace{(a-1)}_{2c-1}$$

$$2^1 \cdot 3 \cdot 5 \cdots p_n$$

10. Prove that it is impossible for three consecutive squares to sum to another perfect squares.

$$x^2 + (x+1)^2 + (x+2)^2$$

$$(x-1)^2 + x^2 + (x+1)^2$$

$$x^2 - 2x + 1 + x^2 + x^2 + 2x + 1$$

$$= 3x^2 + 2 = a^2 = \cancel{9k^2}$$

$$\cancel{3k}$$

$$\cancel{(3k+1)^2 = 9k^2 + 3k + 1}$$

$$\cancel{(3k+2)^2 = 9k^2 + 6k + 4 \sim 3t + 1}$$

11. A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \cdots, 2019\}$ are nice?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5 .

$n = 1 + p + p^2 + p^3$

$1 + 11 + 11^2 + 11^3 < 1800$

$1 + 13 + 13^2 + 13^3 > 2100$

$m = p^3$

$m = p^1 q^1$

$n = 1 + p + q + pq = (1+p)(1+q)$

case 1

$p = 2$

$3(1+q) =$

$2010 = 3 \cdot 670$ ✗

$2016 = 3 \cdot 671$ ✗

case 2

$2012 = 4 \cdot 503$ ✗

$(1+p)(1+q) = 2016 = 4 \cdot 504$ ✓