

The Ninth Grade Math Competition Class

Factorials

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$$(2 \cdot 2 \cdot 2)^n = (2^3)^n = 2^{3n}$$

1. Find the largest integer value of n for which 8^n evenly divides $100!$.

~~$8^1 \frac{100}{8} = 12$~~

~~$8^2 \frac{12}{8} = 1$~~

$$100! = 100 \cdot 99 \cdot \dots \cdot (4 \cdot 3 \cdot 2) \cdot 1$$

$\downarrow \downarrow$
8

$$2^1 \frac{100}{2} = 50$$

$$2^2 \frac{50}{2} = 25$$

$$2^3 \frac{25}{2} = 12$$

$$2^4 \frac{12}{2} = 6$$

$$2^5 \frac{6}{2} = 3$$

$$2^6 \frac{3}{2} = 1$$

$$2^7 \frac{3}{2} = 32$$

2. Find the prime factorization of $10!$.

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1 \end{aligned}$$

3. What is the product of the positive divisors of 7!.

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$

$$7! \cdot 30 = \frac{5 \cdot 3 \cdot 2 \cdot 2}{2} = 30$$

4. How many positive cubes divide $3!5!7!$.

$$3! = 2^1 \cdot 3^1$$

$$5! = 2^3 \cdot 3^1 \cdot 5^1$$

$$7! = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$

$$3!5!7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$$

$$2^0 \quad 3^0 \quad 5^0 \quad 7^0$$

$$2^3 \quad 3^3$$

$$2^6$$

$$3 \cdot 2 \cdot 1 \cdot 1 = 6$$

5. For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 + 2 + \dots + n$?

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

$$(1+n) + (2+n-1) + \dots$$

$$\frac{n(n+1)}{2} \mid n!$$

$$n(n+1) \mid 2n!$$

$$n=6$$

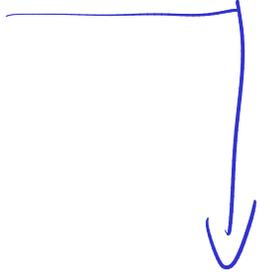
$$6 \cdot 7 \mid 2 \cdot 6! = 2 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n+1 = \cancel{2}, 3, 5, 7, 11, 13, 17, 19, 23$$

$$24 - 8 = 16$$

6. In how many zeros does the decimal expansion of $100^{100} - 100!$ end?

~~1000...0000
124510000
0000~~



$(10^2)^{100}$
 10^{200}

100! end with?

$$5^1 \frac{100}{5} = 20$$

$$5^2 \frac{20}{5} = 4$$

~~24~~

7. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k .

$$P = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots \cdot 199$$

$$P = \frac{200!}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \dots \cdot 200}$$

$$= \frac{200!}{2 \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot (2 \cdot 4) \cdot \dots \cdot 2 \cdot (100)}$$

$$P = \frac{200!}{100! \cdot 2^{100}}$$

$$\begin{array}{l} \left. \begin{array}{l} 3^1 \\ 3^2 \\ 3^3 \\ 3^4 \end{array} \right\} \frac{200}{3} = 66 \\ \frac{66}{3} = 22 \\ \frac{22}{3} = 7 \\ \frac{7}{3} = 2 \end{array}$$

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$$\begin{array}{l} \left. \begin{array}{l} 3^1 \\ 3^2 \\ 3^3 \\ 3^4 \end{array} \right\} \frac{100}{3} = 33 \\ \frac{33}{3} = 11 \\ \frac{11}{3} = 3 \\ \frac{3}{3} = 1 \end{array}$$

48 = (49)