The Ninth Grade Math Competition Class
Factorials
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$(2\cdot 2\cdot 2)^n = (2^3)^n = 2^{3n}$

1. Find the largest integer value of n for which 8^n evenly divides $100!$.

$\cancel{8^1 \quad \frac{100}{8} = 12}$

$\cancel{8^2 \quad \frac{12}{8} = 1}$

$100! = 100\cdot 99 \cdot \ldots \cdot 4 \cdot 3 \cdot 2 \cdot 1$

8

$2^1 \quad \frac{100}{2} = 50$

$2^2 \quad \frac{50}{2} = 25$

$2^3 \quad \frac{25}{2} = 12$

$2^4 \quad \frac{12}{2} = 6$

$2^5 \quad \frac{6}{2} = 3$

$2^6 \quad \frac{3}{2} = 1$

$\frac{97}{3} = 32$

2. Find the prime factorization of $10!$.

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$$

3. What is the product of the positive divisors of 7!.

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$

$$7!^{30} \quad \frac{5 \cdot 3 \cdot 2 \cdot 2}{2} = 30$$

4. How many positive cubes divide $3!5!7!$.

$$3! = 2^1 \cdot 3^1$$

$$5! = 2^3 \cdot 3^1 \cdot 5^1$$

$$7! = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$

$$3!5!7! = 2^8 \, 3^4 \, 5^2 \cdot 7^1$$

$$2^0 \quad 3^0 \quad 5^0 \quad 7^0$$

$$2^3 \quad 3^3$$

$$2^6$$

$$3 \cdot 2 \cdot 1 \cdot 1 = 6$$

5. For how many positive integers n less than or equal to 24 is $n!$ evenly divible by $1 + 2 + \cdots + n$?

$$1+2+\ldots+n-1+n = (n+1)\cdot\frac{n}{2}$$

$$(1+n)+(2+n-1)+\ldots$$

$$\frac{n(n+1)}{2} \mid n!$$

$$n(n+1) \mid 2n!$$

$n=6$

$$6\cdot 7 \mid 2\cdot 6! = 2\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$$

$$n+1 = \cancel{2},\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23$$

$$24-8=16$$

$1000\ldots00000$

$12345\ldots0000$

0000

6. In how many zeros does the decimal expansion of $100^{100} - 100!$ end?

$(10^2)^{100}$

10^{200}

100! end with?

$5^1 \quad \frac{100}{5} = 20$

$5^2 \quad \frac{20}{5} = 4$

24

7. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

$$P = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot 199$$

$$P = \frac{200!}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 200}$$

$$= \frac{200!}{2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 4 \cdots 2 \cdot 100}$$

$$P = \frac{200!}{100! \cdot 2^{100}}$$

3^1 $\frac{200}{3} = 66$

3^2 $\frac{66}{3} = 22$

3^3 $\frac{22}{3} = 7$

3^4 $\frac{7}{3} = 2$

97

3^1 $\frac{100}{3} = 33$

3^2 $\frac{33}{3} = 11$

3^3 $\frac{11}{3} = 3$

3^4 $\frac{3}{3} = 1$

48 $= 49$